

A Theoretical Plethora of Modelling Actuarial Risk Aversion Coefficient

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Abstract: *The goal of the paper is to theoretically evaluate an approximate actuarial aversion risk coefficient in relation to future utility trend and discuss an analytic model for investigating the behaviour of risk aversion random risk together with the influence it exerts on utility function. By initiating Newton's process, the result shows that the scheme holder's risk premium for small actuarially neutral risk, is the product of half of the aversion and the volatility term. The paper stresses the importance of numerical methods in actuarial risk theory and also brings our attention to risk measurement applications. Furthermore, it describes the procedure of estimating the intensity of aversion co-efficient using numerical algorithm. It relies heavily on the analytic properties of utility function whose gradient function does not vanish. The estimation of aversion coefficient lends credence to risk theory because of its potency to measure riskiness of insurance portfolio guiding both risk manager and scheme holder either or not to assume risk. However, the estimation of aversion involves a model based on the knowledge of differential equation.*

Keywords: Risk aversion, Functional, Equilibrium, Integral, Utility

1. Introduction

The objective of this paper is to obtain particular views of utility function and its relevance in insurance applications in particular to estimate the co-efficient of risk aversion, the maximum premium for a scheme holder buying an insurance product and the utility gradient function using the Taylor's series expansion to the appropriate degree of accuracy. When the utility of wealth $u(y)$ is required at different levels and underlying mathematical formula for risk aversion is not given, then the aversion $a(y)$ can only be determined by estimation. Underwriters are not usually ready to assume certain risks due to the nature of complexity involved in underwriting process. The problem of estimating risk aversion co-efficient at any given instant occurs frequently in insurance risk since it

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measures the degree of unwillingness of the insurers to assume underwriting risk. O'Donoghue and Jason (2018) found that in a bid to capturing aversion risk co-efficient, the benchmark in actuarial risk theory is to invoke the expected utility model in which aversion risk co-efficient is obtained from diminishing marginal utility of wealth. If $u(y)$ denotes expected utility of wealth at a level y and $a(y)$ is the instantaneous aversion coefficient, then it is not possible to evaluate the value of $a(y)$ analytically from the second order ordinary differential equation described by $a(y) = -\frac{u''(y)}{u'(y)}$ unless $u(x+t)$

can be functionally re-expressed as a convergent series polynomial function. This is a mathematical framework for studying the relationship between the gradient function of expected utility of wealth at a level y , instantaneous risk aversion co-efficient and the marginal utility function. On the basis of known levels of wealth, the task is to accurately estimate aversion coefficient $a(y)$ at any instant using Taylor's series expansion with an assumption that $u(y)$ is a convergent series polynomial function. Given the utility function $u(y)$, a second problem is how to measure approximately to maximum premium Σ^+ for a random risk Y . Therefore, the task is to establish approximate value instead of their real analytical value using tools of approximation. The aversion $a(y)$ can now be approximated from the numerical point of view by invoking limiting processes so that inference can be drawn about the probability of an insurer not taking up a risk in an interval of time. The estimation of $a(y)$ is by far an interesting but a difficult problem since part of the difficulty is the computation of maximum premium Σ^+ that an insured will be ready to pay for a risk if the only information given is $u(y)$. The aversion coefficient as observed in (Pratt, 1964) is subject to various interpretation as a measure of local risk aversion such that both the curvature of $u(y)$ and $u''(y)$ may not sufficiently measure the aversion. The value of $u'(y)$ at a point on the u curve is the gradient or the direction at that point.

2. The Curvature of the Utility Function

The degree of aversion to risk is dependent on the curvature of the underlying utility function, it is logical to measure it with second derivative. Aversion coefficient function is characterized by invariance under linear transformations of $u(w)$ as can be seen.

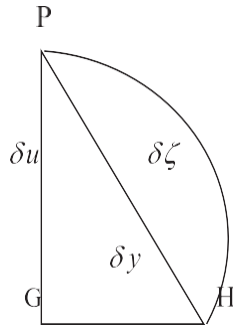
$$\left. \begin{aligned} U &= C + ku(w) \\ U'(w) &= ku'(w) \\ U''(w) &= ku''(w) \end{aligned} \right\}$$

Then $\frac{U''(w)}{U'(w)} = \frac{u''(w)}{u'(w)}$. And because of this invariance property, the second derivative $u''(w)$ is not an efficient measure of aversion to risk and consequently, the second derivative is normalized to measure the aversion to risk adequately. Increasing or decreasing absolute or

relative risk aversion may have consequential implications on portfolio having some risky asset and some risk-free asset. If an insurer experiences an increase in asset, then he can increase the value of insurance portfolio held in risky asset if absolute value of aversion falls.

$$a(w) = wA'(w), a'(w) < 0 \Rightarrow A'(w) < 0$$

The aversion to risk is a measure of concavity associated with the aversion coefficient, therefore aversion $a(y)$ measures concavity of the utility at a level of asset y . The only available option to measure concavity is either $u''(y)$ or by the curvature of $u(y)$. The curvature of the utility function measures how rapid the utility function changes tangential direction in the neighborhood of the defined point. In literature, curvature is measured by the radius of curvature Γ .



In order to analyze the curvature, the method of geometry is used. We let H, P, G be a right angled triangle right angled at G whose small hypotenuse, length $\delta\zeta$ is representing the length of the arc from G to P and such that $w(\zeta)$ is the gradient at H . The height PG is δu while base HG is δy . Because of the infinitesimally small size $\delta\zeta$ of the hypotenuse, the direction of the chord from H to P and the direction of the tangent to the arc running from H to P will be very close as their numerical difference approaches value zero. As $\delta\zeta$ approaches zero,

$$\frac{du}{dy} = \tan \omega(\zeta).$$

$$\frac{dy}{d\zeta} = \cos \omega(\zeta)$$

$$\frac{d}{d\zeta} \left(\frac{du}{dy} \right) = \frac{d \tan \omega(\zeta)}{d\zeta}$$

$$\frac{d}{dy} \left(\frac{du}{dy} \right) \left(\frac{dy}{d\zeta} \right) = \frac{d}{d\omega} (\tan \omega) \frac{d\omega(\zeta)}{d\zeta}$$

$$\cos \omega(\zeta) \left(\frac{d^2 u}{dy^2} \right) = \sec^2 \omega(\zeta) \frac{d\omega(\zeta)}{d\zeta}, \text{ hence}$$

$$\left(\frac{d^2u}{dy^2}\right) = \sec^3 \omega(\zeta) \frac{d\omega(\zeta)}{d\zeta}$$

$$\sec^3 \omega(\zeta) = (\sec^2 \omega(\zeta))^{\frac{3}{2}} = [\tan^2 \omega(\zeta) + 1]^{\frac{3}{2}} = \left[\left(\frac{du}{dy}\right)^2 + 1\right]^{\frac{3}{2}}$$

$$\frac{d\omega(\zeta)}{d\zeta} = \frac{1}{\rho} = \frac{\frac{d^2u}{dy^2}}{\left[\left(\frac{du}{dy}\right)^2 + 1\right]^{\frac{3}{2}}} = \frac{\sec^3 \omega(\zeta) \frac{d\omega(\zeta)}{d\zeta}}{\left[\left(\frac{du}{dy}\right)^2 + 1\right]^{\frac{3}{2}}}$$

$$\rho = \frac{\left[\left(\frac{du}{dy}\right)^2 + 1\right]^{\frac{3}{2}}}{\frac{d^2u}{dy^2}} = \frac{d\zeta}{d\omega}$$

If the utility function $u(y)$ is given, then the first and the second derivatives can be computed to obtain the curvature. The expected utility model representing the canonical theory of choice under risk uncertainty functionally describes the financial health of insurer where an insured is risk averse and as a rational policy holder, inadvertently pays in excess of the expected value of his claims to ensure coverage. The set conditions through which decisions are taken under uncertainty is by comparing the expected utilities associated with these payoffs.

3. Literature Review

Insurance contracts are the transfer mechanism of the risk Y from policyholder to the underwriter where compensation is paid in form of indemnity which is not determined in advance by either of the insured or insurance company. In Kaas et al. (2008); Liu and Tsai-Ling (2021), an insurance policy is be defined by an indemnity function $\Sigma: R^+ \rightarrow R^+$ however, it is noted that hardly does an insurance policy fully cover the whole of insured peril, so that the risk manager cover only a threshold limit $\Sigma(y)$ while the policy holder retains the remaining portion of the scheme $S(y) = [Y - \Sigma(y)]$, S is the threshold limit. provided, $Y > \Sigma(y)$ so that in the event the random loss is smaller than the threshold limit defined by insurer's proportion, the insurance company may legally repudiate claim filed by the insured and hence will not be liable to indemnity the insured as attested to in the terms and condition of the policy document. Kaas et al. (2008) agreed that in order to drastically reduce the financial impact of extreme random

losses which could account for negative premium and simultaneously constraint the insured from making any gain from the contract, the indemnity function $\Sigma(y)$ must be bounded in a domain $D \subset R^+$ such that $0 < \Sigma(y) < y$ where the lower bound suggests that $\Sigma(y)$ is a positive random function while the upper bound is the limit of indemnity. In Eisenhauer and Halek (1999); Eeckhoudt (2012); Johannes et al. (2018), the insured pays a premium p in exchange for transferred risk to the insurer and then the insured policy is defined as the ordered pair $(p, \Sigma(y))$ whose initial wealth w now declines to $[w + \Sigma(y) - (p + y)]$. It is on this basis that the utility model establishes that a rational policy holder who is risk averse is ready to pay the maximum premium more than the expected value of his claims $E(Y)$ to ensure he is covered. A risk averse takes an insurance decision under uncertainty by weighing the expectation of the utility function on Jensen's inequality scale. Given a sequence of random losses $Y_1, Y_2, \dots, Y_{n-2}, Y_{n-1}, Y_n$, The insured takes the mathematical expectation of

$$E(U(W - Y_1)), E(U(W - Y_2)), E(U(W - Y_{n-1}))$$

In order to reduce his risk, he can numerically evaluate which expected utility has the highest value so that he can choose the appropriate loss. It is inferred from Buhlmann (1970); Buhlmann and Jewell (1979); Buhlmann (1984); Gerber and Shiu (1994); Gerber and Shiu (1996); Kaas et al. (2008) that the condition of equivalent expected utility is applicable to replace random gain by a constant amount and to compute actuarially fair premium for insuring the risk, even if the insurer's wealth is not a random function. The consideration is actuarially fair as a function of its expected utility if it comprises of a loading factor depending on the insurer's risk aversion co-efficient together with the joint distribution of the claims and the random wealth. Consequently, an insured with wealth w can functionally determine the maximum premium p , he is prepared to pay for a random loss.

$Y_i, i = 1, 2, 3, 4, \dots, n$ or he can compare the variances

$$\sigma_{Y_1}^2, \sigma_{Y_2}^2, \dots, \sigma_{Y_{n-2}}^2, \sigma_{Y_{n-1}}^2, \sigma_{Y_n}^2 \quad \exists: \text{if } \sigma_{Y_1}^2 < \sigma_{Y_2}^2 < \dots < \sigma_{Y_{n-2}}^2 < \sigma_{Y_{n-1}}^2 < \sigma_{Y_n}^2$$

Then it is apparent that the insured prefers the insurance with random loss Y_1 , with the lowest variance provided $E(Y_1) = E(Y_2) = \dots = E(Y_n)$. The maximum premium $\max P$, the policy holder can afford to pay is obtainable through the application of the utility functional on expressions $(w - Y)$ and $(w - P)$ and then taking the expectation of the first. The two expressions of the utility function $U(\cdot)$ when equated to zero will be the equilibrium equation. Lienhard (1986); Kaas et al. (2008) observed that when considering a random loss Y , the equilibrium equation for the insured becomes.

$$E[u(w-Y)] - u(w - \Sigma(P)) = 0 \quad (1)$$

$$E[u(w-Y)] = u(w - \Sigma(P)) \quad (2)$$

where $\Sigma(P)$ is the premium the insured is ready to pay. The maximum premium Σ^+ for which equation (1) holds represents the zero premium because it is the basis premium which satisfies equilibrium condition. At equilibrium level, the scheme holder is indifferent whether or not his risk is insured. The insurer charges minimum premium for insuring random loss Y and analogous to the insured, the two expressions $E[u(W + P - X)]$ and $u(W)$ form the basis for the equilibrium equation for the insurer.

$$E[u(W + \Sigma(P) - Y)] - u(W) = 0 \quad (3)$$

$$E[u(W + \Sigma(P) - Y)] = u(W) \quad (4)$$

The value of minimum premium $\Sigma(P) = \Sigma^-$ which satisfies the insurer's equation is called zero premium. In equation (3), if the condition $\Sigma^- < \Sigma(P) < \Sigma^+$ holds, then the utility function $u(\cdot)$ of the scheme holder and insurer must increase since utility function increases with increasing wealth for some class of utility function. Since $u(\cdot)$ is a non-decreasing function, a risk averse policy holder will prefer a fixed loss such as ordinary deductible to a random loss having same expected utility. Experts have formulated a few models to guide how scheme holders taking decisions can select between uncertain events. If the decision maker steadily chooses among potential random losses Y , then there exists a utility function $u(\cdot)$ attached to his wealth w such that the insurance decisions he takes are exactly the same as those resulting from comparing the losses Y based on the expectation $E[u(w - Y)]$ and consequently a complex decision would have been reduced in intensity by comparison. For the comparison of losses X with Y , the utility function $u(t)$ and its linear transform $au(t) + b, a > 0$ are equivalent since they result in the same decision.

3.1 Theorem

If X, Y are random losses, then

$$E[u(w - X)] \leq E[u(w - Y)] \quad \text{if} \quad E[au(w - X) + b] \leq E[au(w - Y) + b] \quad (5)$$

3.1.1 Proof

We note that if $u(y) = c + dy$ for all y , then we have

$$(Eu(w+h)) = E[c+d(w+h)] = c+d(w+E(h)) \tag{6}$$

$$E[c+d(w+h)] = c+d(w+E(h)) = u(w+E(h)) \tag{7}$$

This means that the policy holder ranks insurance based on his expected outcome. The attitude of this policy holder is said to be risk-neutral.

$$\text{It suffices to proof that if } E[au(w-X)+b] \leq E[au(w-Y)+b] \tag{8}$$

$$\text{Then } E[u(w-X)] \leq E[u(w-Y)] \tag{9}$$

Recall that a twice-differentiable function $u(w-y)$ is convex on an interval (c,d) if and only if $u''(w-y) \geq 0$ for all $(w-y)$ in (c,d) (10)

Applying Taylor's on both sides of $au(w-X)+b = au(w-Y)+b$, we have

$$b+a\frac{(w-x)^1}{1!}u(b)+a\frac{(w-x)^2}{2!}u^{(2)}(b) \leq b+a\frac{(w-y)^1}{1!}u(b)+a\frac{(w-y)^2}{2!}u^{(2)}(b) \Rightarrow \tag{11}$$

$$a\frac{(w-x)^1}{1!}u(b)+a\frac{(w-x)^2}{2!}u^{(2)}(b) \leq a\frac{(w-y)^1}{1!}u(b)+a\frac{(w-y)^2}{2!}u^{(2)}(b) \Rightarrow \tag{12}$$

$$\frac{(w-x)^1}{1!}u(b)+\frac{(w-x)^2}{2!}u^{(2)}(b) \leq \frac{(w-y)^1}{1!}u(b)+\frac{(w-y)^2}{2!}u^{(2)}(b) \tag{13}$$

$$u(w-x)u(b)+u\left(\frac{(w-x)^2}{2!}\right)u^{(2)}(b) \leq u(w-y)^1+u\left(\frac{(w-y)^2}{2!}\right)u^{(2)}(b) \tag{14}$$

$$\text{either, } u(w-x)u(b) \leq u(w-y)u(b) \text{ or } u\left(\frac{(w-x)^2}{2!}\right)u^{(2)}(b) \leq u\left(\frac{(w-y)^2}{2!}\right)u^{(2)}(b) \tag{15}$$

$$u(w-x)u(b) \leq u(w-y)u(b) \Rightarrow u(w-x) \leq u(w-y) \tag{16}$$

now taking mathematical expectations

$$Eu(w-x) \leq Eu(w-y) \tag{17}$$

The implication is that the expectations function $E(.)$ function is a coherent risk measure on random losses X, Y . In general, if $X_1, X_2, X_3, X_4, \dots, X_{n-1}, X_n$ is a sequence of random losses, then it follows that

$$E[u(w-X_1)] \leq E[u(w-X_2)] \leq E[u(w-X_3)] \leq E[u(w-X_{n-1})] \dots \leq E[u(w-X_n)] \tag{18}$$

The theory of equivalent expected utility could substitute for random gain at a constant value in order to obtain actuarially fair premium on claims meant to be covered even when the insurer's wealth (without the new scheme) is a random function.

3.2 Utility Function

The function $u(\cdot): A \rightarrow R$ attaches a value to each possible wealth where A is an open interval and R is a set of real numbers. It is assumed that the gradient function is increasing that is $u'(y) > 0$ and concave, where in the relative value of a unit currency decreases as y increases $u''(y) < 0$. As with other continuously differentiable functions, it is assumed that $u(y)$ possesses differential coefficients of all orders at the points at which it is defined and hence possess Taylor series expansion about a regular point of analyticity.

$u(y+t)$ will be analytic if it has a Taylor's series expansion converging to $u(y)$. The Taylor's concerns approximation of sufficiently smooth function $u(y)$ by polynomials in a neighbourhood of a particular level of wealth w . The domain A of monetary outcomes is either the whole of real line or the positive real line. We assume that an underwriter has wealth w . with utility function $U(\cdot)$. We also assume that the insurer has the choice to cover a risk G . the insurance firm has two choices (i) it may reject the risk based on underwriting results or it accepts the risk but charges a premium Σ . The two cases above correspond to the preference order where Σ is governed by $U(W) \leq E(U(\Sigma - G + W))$ (21)

Consequently, the condition $U(W) = E(U(\Sigma - G + W))$ (22)

is valid when the minimum premium Σ^- is considered. This minimum premium is characterized in such a way that the underwriter is indifferent between accepting or rejecting the insurance risk hence the equality conditions is the principle of equivalent utility and consequently, the zero premium which solves the equation is the indifference premium. let $\Sigma = \Sigma_z$ (23)

be the zero of the equation above, that is, the indifference premium, then by Jensen's inequality

$$E(U(\Sigma - G + X_0)) \leq U(\Sigma_z - E(G) + X_0) \quad (24)$$

if $U(\cdot)$ is strictly increasing then $\Sigma_z - E(G) \geq 0$ (25)

hence $\Sigma_z \geq E(G)$, We infer from Gajura (2010), that an acceptable insurance premium for any insurance policy is a premium associated to that underwriter with preferences towards risk and wealth defined by his utility function $U(\cdot)$ and consequently distinct underwriters would impose varying premiums based on the underwriting experience and result for covering the risk G .

Following Arrow (1963); Pratt (1964); Arrow (1970), underwriters' or insureds' behaviour to risk should fall in line with absolute risk aversion. It is assumed that $u(y)$ is a function of total assets of an insurance firm so that

$y = 0$ either results in total loss or insolvency.

3.3 Correlation in Mean Variance of Two Random Risks

Let Y, X be two random risks such that $Y = cX + d$ (26)

$E(X) = X - x$ and $E(Y) = Y - y, E(X) = \mu_x, E(Y) = \mu_y$ (27)

Therefore $\left(y - \rho \frac{\sigma_Y}{\sigma_X} x\right)^2 = y^2 + \rho^2 x^2 \frac{\sigma_Y^2}{\sigma_X^2} - 2x\rho \frac{\sigma_Y}{\sigma_X} y$ (28)

And taking expectation, hence

$E\left(y - \rho \frac{\sigma_Y}{\sigma_X} x\right)^2 = E\left(y^2 + \rho^2 x^2 \frac{\sigma_Y^2}{\sigma_X^2} - 2\rho \frac{\sigma_Y}{\sigma_X} xy\right) =$ (29)

$E\left((Y - E(Y))^2 + \rho^2 x^2 \frac{\sigma_Y^2}{\sigma_X^2} - 2\rho \frac{\sigma_Y}{\sigma_X} xy\right)$

$E(Y^2) + [E(Y)]^2 - 2[E(Y)]^2 + \rho^2 \frac{\sigma_Y^2}{\sigma_X^2} (E(X^2) + [E(X)]^2 - 2[E(X)]^2) -$ (30)

$2 \frac{\sigma_Y}{\sigma_X} \rho [E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)]$

$E\left(y - \rho \frac{\sigma_Y}{\sigma_X} x\right)^2 = E(Y^2) - \mu_y^2 + \rho^2 \frac{\sigma_Y^2}{\sigma_X^2} (E(X^2) - \mu_x^2) - 2\rho \frac{\sigma_Y}{\sigma_X} (E(XY) - \mu_x \mu_y)$ (31)

$E\left(x - \rho \frac{\sigma_Y}{\sigma_X} y\right)^2 = \sigma_y^2 + \rho^2 \frac{\sigma_Y^2}{\sigma_X^2} \sigma_x^2 - 2\rho \frac{\sigma_Y}{\sigma_X} COV(X, Y)$ (32)

$E\left(x - \rho \frac{\sigma_Y}{\sigma_X} y\right)^2 = \sigma_y^2 [1 + \rho^2] - 2\rho \frac{\sigma_Y}{\sigma_X} COV(X, Y)$ (33)

$COV(X, Y) = (E(XY) - \mu_x \mu_y) = \sigma_x \sigma_y \rho$ (34)

$E\left(x - \rho \frac{\sigma_Y}{\sigma_X} y\right)^2 = \sigma_y^2 [1 + \rho^2] - 2 \frac{\sigma_Y}{\sigma_X} \sigma_x \sigma_y \rho^2$ (35)

$\sigma_y^2 + \sigma_y^2 \rho^2 - 2\sigma_y^2 \rho^2 = \sigma_y^2 - \sigma_y^2 \rho^2 = \sigma_y^2 (1 - \rho^2)$ (36)

Setting $X = x, Y = y$ (37)

Then $\mu_y = \mu_x = 0$ so that $COV(X, Y) = \sigma_x \sigma_y \rho$ (38)

If $\rho = 1$ then $E\left(x - \rho \frac{\sigma_Y}{\sigma_X} y\right)^2 = 0$ and $x - \rho \frac{\sigma_Y}{\sigma_X} y = 0 \Rightarrow x = \rho \frac{\sigma_Y}{\sigma_X} y$ (39)

$$\text{Using equation (39) } \sigma_X(X - \mu_X) = \rho\sigma_Y(Y - \mu_Y) \quad (40)$$

$$\frac{(X - \mu_X)}{(Y - \mu_Y)} = \frac{\rho\sigma_Y}{\sigma_X} \text{ and } \frac{x}{y} = \frac{\rho\sigma_Y}{\sigma_X} \quad (41)$$

$$Y = CX + d \Rightarrow \mu_Y + y = C(\mu_X + x) + d \quad (42)$$

$$y = C\mu_X - \mu_Y + d + Cx \quad (43)$$

4. Materials and Methods

In this study, the Taylor's series technique was adopted in constructing aversion risk coefficient as demonstrated in equations 11-16 and equations 44-74.

Following Elabed and Michael (2015), a critical concern of actuarial risk theory is how to evaluate the risk of the uncertainty in the future value of an insurance portfolio. This evaluation is usually achievable by modeling the uncertainty in $u(\cdot)$ as a random variable to which a certain risk measure is applied. The risk measure describes a single value which is reasonable to provide a sufficient measure of size of the level of risk exposure. Given a risk functional $u(\cdot): A \rightarrow R$, which assigns a wealth to a rational scheme holder then any of the following relations occurs.

$u''(x) < 0; u''(x) = 0(1); u''(x) > 0$ depending on whether the policy holder is risk averse or risk neutral or risk loving if his marginal function is less than zero, equal to zero and greater than zero respectively, where $0(1)$ is any function that is vanishingly zero.

Following Eeckhoudt et al. (2016); Eeckhoudt et al. (2018), the risk aversion as measured by risk aversion coefficient is the unwillingness of the insurer to assume risk in an interval of time irrespective of the nature and volume of the insurance scheme. When the risk aversion co-efficient increases, the premium which the insured is willing to pay increases. Therefore, when there is high degree of aversion intensity, the insurer will not be willing to guarantee coverage. If Y is a loss random variable then $a(y)$ may be inferred as the riskiness of an insurance portfolio or the amount of capital that should be added to a portfolio with a loss $a(y)$ so that the insurance portfolio can be assessed acceptable from a risk point of view. An insurer's sense of judgment to sell an insurance policy to the insuring public at a premium Σ would offer the insurance firm a stochastic variable profit margin. The probability distribution of the random variable functionally depends on Σ , claim distribution and demand function. The insurance regulation which assists the underwriter to fix a premium at which a scheme is to be sold is same regulation which guides to identify and select the best profit distributions and hence shows the willingness to assume risky policy. The selection of a market premium Σ implies and is implied by selection of a profit distribution. In order to analytically construct the aversion, certain conditions are required, for instance we assume the

existence of a continuous risk aversion function $a(y)$ whose values at wealth level y furnish us a close approximation to the degree of unwillingness of insurers to take up risk in the interval $y \rightarrow (y+t)$ since instantaneous rate $u'(y)$ is involved, it usually takes the form of a derivative and limiting process. Let the mean and variance of random loss Y be μ_Y and σ^2_Y respectively. The utility function is characterized as follows as observed in Klugman et al. (2004); Kaas et al. (2008); Guo et al. (2016):

- (i) $u(w)$ is an increasing function of w that is $u'(w) > 0$ where the gradient function is positive and since insurer or insured prefers more to less.
- (ii) $u(w)$ is a concave function of w that is $u''(w) < 0$ where it is required that the marginal utility function of $u(w)$ is a decreasing function of wealth w and that the utility function $u(w)$ is twice differentiable the reason being that insured or insurer is risk averse

$$au(w - \Sigma^+) + b = au[(w - \mu_Y) + (\mu_Y - \Sigma^+)] + b \quad (44)$$
- (iii) The utility gradient function is not vanishingly zero that is $u'(y) \neq 0(1)$ otherwise the aversion co-efficient will be unbounded
- (iv) We impose that $u(.)$ necessarily possess derivatives of all orders

It is important that $U(.)$ must be bounded and sufficiently regular or smooth to justify the proof of risk aversion

$$w - \Sigma^+ = (w - \mu_Y) + (\mu_Y - \Sigma^+) \quad (45)$$

Taking the utility of both sides

$$au(w - \Sigma^+) + b = au[(w - \mu_Y) + (\mu_Y - \Sigma^+)] + b \quad (46)$$

$$\text{Similarly, } (w - Y) = (w - \mu_Y) + (\mu_Y - Y) \quad (47)$$

$$au(w - Y) + b = au[(w - \mu_Y) + (\mu_Y - Y)] + b \quad (48)$$

$$\bar{Y} = (\mu_Y - Y) = \text{risk neutral since } E(\bar{Y}) = 0 \quad (49)$$

w = wealth or asset of policy holder or insurer

Σ^+ = maximum premium

$$(\Sigma^+, \mu_Y) = \text{risk premium} \quad (50)$$

$$E(w - Y) - (w - \Sigma^+) = E(w) - E(Y) - w + \Sigma^+ = \quad (51)$$

$$w - \mu_Y - w + \Sigma^+ = \Sigma^+ - \mu_Y$$

$$(\Sigma^+, \mu_Y) \text{ must satisfy the } Eu(w + \bar{Y}) = u(w - \pi) \quad (52)$$

Applying Taylor's series expansion of order one to equation (46) and Taylor's series expansion of order two to equation (45), we have,

$$\frac{u[(w - \mu_Y) + (\mu_Y - \Sigma^+)] - u(w - \mu_Y)}{(\mu_Y - \Sigma^+)} = u'(w - \mu_Y) + \quad (53)$$

$$\frac{(\mu_Y - \Sigma^+)}{2} u''(w - \mu_Y) + o(\mu_Y - \Sigma^+)^2$$

$$\Rightarrow \frac{\Delta u(w - \mu_Y)}{(\mu_Y - \Sigma^+)} = \frac{du}{d(\mu_Y - \Sigma^+)} + o(\mu_Y - \Sigma^+) \quad (54)$$

Where the $o(\mu_Y - \Sigma^+)$ means that the first term neglected is of order $(\mu_Y - \Sigma^+)$

$$u[(w - \mu_Y) + (\mu_Y - \Sigma^+)] = (\mu_Y - \Sigma^+) u'(w - \mu_Y) + \frac{(\mu_Y - \Sigma^+)^2}{2} u''(w - \mu_Y) + \quad (54a)$$

$$o(\mu_Y - \Sigma^+)^2$$

$$au(w - \Sigma^+) + b = \left\{ \frac{au(w - \mu_Y)(\mu_Y - \Sigma^+)^0}{0!} + \frac{au'(w - \mu_Y)(\mu_Y - \Sigma^+)^1}{1!} + \right. \quad (55)$$

$$\left. \frac{au''(w - \mu_Y)(\mu_Y - \Sigma^+)^2}{2!} + b \right\}$$

$$au(w - \Sigma^+) + b = \{ au(w - \Sigma^+) + b \cong au(w - \mu_Y) + \quad (56)$$

$$au'(w - \mu_Y)(\mu_Y - \Sigma^+)^1 + b \}$$

Using first order

$$au(w - Y) + b = au(w - \mu_Y) + au'(w - \mu_Y)(\mu_Y - Y) + \quad (57)$$

$$\frac{au''(w - \mu_Y)(\mu_Y - Y)^2}{2} + b$$

Recall $aE[u(w - Y)] + b = au(w - \Sigma^+) + b$, now taking the mathematical expectation of left-hand side alone to obtain.

$$aE[u(w-Y)]+b = aE\left[u(w-\mu_Y)+u'(w-\mu_Y)(\mu_Y-Y)+\frac{u''(w-\mu_Y)(\mu_Y-Y)^2}{2}\right]+b \quad (58)$$

Then

$$aE[u(w-Y)]+b = aE[u(w-\mu_Y)]+aE[u'(w-\mu_Y)(\mu_Y-Y)]+aE\left[\frac{u''(w-\mu_Y)(\mu_Y-Y)^2}{2}\right]+b \quad (59)$$

$$\Rightarrow aE[u(w-Y)]+b = aE[u(w-\mu_Y)]+aE[u'(w-\mu_Y)E(\mu_Y-Y)]+aE\left[\frac{u''(w-\mu_Y)(Y-\mu_Y)^2}{2}\right]+b \quad (60)$$

$$aE[u(w-Y)] = aE[u(w-\mu_Y)]+au'(w-\mu_Y)[E(Y)-\mu_Y]+\frac{au''(w-\mu_Y)E(Y-\mu_Y)^2}{2!}+\frac{au'''(w-\mu_Y)E(Y-\mu_Y)^3}{3!}+\frac{au^{iv}(w-\mu_Y)E(Y-\mu_Y)^4}{4!} \quad (61)$$

$$aE[u(w-Y)] = aE[u(w-\mu_Y)]+au'(w-\mu_Y)[E(Y)-\mu_Y]+\frac{1}{2}au''(w-\mu_Y)\int_{-\infty}^{\infty}(y-E(y))^2 f_Y(y)dy+\frac{1}{3!}au'''(w-\mu_Y)\int_{-\infty}^{\infty}(y-E(y))^3 f_Y(y)dy+\frac{1}{4!}au^{iv}(w-\mu_Y)\int_{-\infty}^{\infty}(y-E(y))^4 f_Y(y)dy \quad (62)$$

Where $\mu_n = \int_{-\infty}^{\infty}(y-E(y))^n f_Y(y)dy$ and $\mu_2 = \int_{-\infty}^{\infty}(y-E(y))^2 f_Y(y)dy = \sigma^2_Y$

$\mu_2 = \int_{-\infty}^{\infty}(y-E(y))^2 f_Y(y)dy = \sigma^2_Y$ is the second central moment

$$aE[u(w-Y)] = aE[u(w-\mu_Y)]+au'(w-\mu_Y)[E(Y)-\mu_Y]+\frac{au''(w-\mu_Y)\sigma^2_Y}{2}+\frac{1}{3!}au'''(w-\mu_Y)\mu_3+\frac{1}{4!}au^{iv}(w-\mu_Y)\mu_4 \quad (63)$$

$$aE[u(w-Y)] = aE[u(w-\mu_Y)]+au'(w-\mu_Y)[E(Y)-\mu_Y]+\frac{au''(w-\mu_Y)\sigma^2_Y}{2}+\frac{\sigma^3_Y}{3!}au'''(w-\mu_Y)\frac{\mu_3}{\sigma^3_Y}+\frac{\mu_4\sigma^4_Y}{(\mu_4-3\sigma^4_Y)4!}au^{iv}(w-\mu_Y)\left(\frac{\mu_4}{\sigma^4_Y}-3\right) \quad (64)$$

$$aE[u(w-Y)] = aE[u(w-\mu_Y)]+au'(w-\mu_Y)[E(Y)-\mu_Y]+\frac{au''(w-\mu_Y)\sigma^2_Y}{2}+\frac{\sigma^3_Y}{3!}au'''(w-\mu_Y)\gamma_1+\frac{\mu_4}{\gamma_2 4!}au^{iv}(w-\mu_Y)\gamma_2 \quad (65)$$

Where γ_1 and γ_2 are co-efficient of skewness and kurtosis

$$aE[u(w-Y)] = aE[u(w-\mu_Y)] + au'(w-\mu_Y)(\mu_Y - \mu_Y)] + \frac{au''(w-\mu_Y)\mu_2}{2} \quad (66)$$

$$aE[u(w-Y)] = aE[u(w-\mu_Y)] + \frac{au''(w-\mu_Y)\mu_2}{2} \quad (67)$$

$$aE[u(w-Y)] = au(w-\mu_Y) + \frac{au''(w-\mu_Y)\mu_2}{2} \quad (68)$$

$$\text{Recall } u(w-\Sigma^+) = u(w-\mu_Y) + u'(w-\mu_Y)(\mu_Y - \Sigma^+) \quad (69)$$

$$u(w-\mu_Y) + \frac{u''(w-\mu_Y)\mu_2}{2} = u(w-\mu_Y) + u'(w-\mu_Y)(\mu_Y - \Sigma^+) \Rightarrow \quad (70)$$

$$u'(w-\mu_Y)(\mu_Y - \Sigma^+) = \frac{u''(w-\mu_Y)\mu_2}{2} \quad (71)$$

$$(\mu_Y - \Sigma^+) = \left[\frac{\mu_2 u''(w-\mu_Y)}{2u'(w-\mu_Y)} \right] \Rightarrow \Sigma^+ = \mu_Y - \frac{\mu_2 u''(w-\mu_Y)}{2u'(w-\mu_Y)} \quad (72)$$

$$\Sigma^+ = \mu_Y - \frac{\mu_2}{2} a(w, \mu_Y) \quad (73)$$

It is expected that the variance will be infinitesimally small to be extent that $\mu_2 \rightarrow 0$ and consequently $\Sigma^+ = \mu_Y$. The policy holder's risk premium for small actuarially neutral risk Y , is the product of half of the aversion $a(y)$ and the volatility term μ_2

$$\text{From equation (71), } 2(\Sigma^+ - \mu_Y) = \left[-\frac{\mu_2 u''(w-\mu_Y)}{u'(w-\mu_Y)} \right]$$

According to Pratt (1964), the aversion to risk is twice the risk premium per unit of variance for all infinitesimal risk. The expression $\frac{\mu_2}{2}$ represents the volatility term because it is an incoherent risk measure of variability while its coefficient term $\left[-\frac{u''(w-\mu_Y)}{u'(w-\mu_Y)} \right]$ defines the risk aversion coefficient in the form

$$a(w, \mu_Y) = -\frac{u''(w-\mu_Y)}{u'(w-\mu_Y)} \quad (74)$$

Following Szpiro (1986); Eisenhauer and Halek (1999); Kaas et al. (2008); Chiappori and Paiella, (2011); Zhang (2017), we set $\mu_Y = 0$. Then the percentage change in marginal utility per unit of outcome space:

$$\text{Then } a(w) = -\frac{u''(w)}{u'(w)} = -\frac{\frac{d}{dw}\left(\frac{du}{dw}\right)}{\frac{du}{dw}} = -\frac{\frac{d}{dw}(u'(w))}{u'(w)} = -\frac{d}{dw} \ln\left(\frac{du}{dw}\right) = -\frac{\left(\frac{du'(w)}{u'(w)}\right)}{dw} \quad (75)$$

In Elabed and Michael (2015); Ogungbenle (2019); Ogungbenle and Ihedioha (2019), we see that the coefficient of risk aversion defines the inverse quotient of second order derivative of utility function to the first order derivative of utility function.

$A(w) = \frac{a(w)}{w}$ is the relative aversion to risk. Under aversion to risk dynamics, $a(w)$ is often positive, however it may assume zero and negative value for risk-neutral and risk-loving insurance agent.

5. Discussion of Results

The contributions of this paper are based on the evaluation of the following key result areas which are obtained by considering the effect of aversion risk utility model.

(i) Suppose an insurer is given the aversion co-efficient but he is intended is to obtain the corresponding utility function, then we solve the equation

$$\begin{aligned} \frac{d^2u}{dw^2} + a(w)\frac{du}{dw} &= 0 \\ u(\zeta) &= 0 \\ u'(\zeta) &= 1 \end{aligned} \quad (76)$$

Then following Ogungbenle and Ihedioha (2019),

$$u(w) = \int_{\zeta}^w e^{-\int_{\zeta}^{\theta} a(\theta) d\theta} dT \quad (76a)$$

(ii) Suppose the risk neutrality $w = ky + \alpha$ is a linear transformation in equation (75) where k and α are real constants, then

$$\frac{da}{dw} = \frac{da}{dy} \frac{dy}{dw} = \frac{1}{k} \frac{da}{dw} \Rightarrow \frac{d^2a}{dw^2} = \frac{1}{k} \frac{d^2a}{dy^2} \frac{dy}{dw} = \frac{1}{k^2} \frac{d^2a}{dw^2} \quad (76b)$$

$$\frac{d^2a}{dw^2} = \frac{1}{k} \frac{d^2a}{dy^2} \frac{dy}{dw} = \frac{1}{k^2} \frac{d^2a}{dw^2} \quad (76c)$$

$$a(w) = -\frac{u''(w)}{u'(w)} = -\frac{\frac{1}{k^2} \frac{d^2a}{dw^2}}{\frac{1}{k} \frac{da}{dw}} = -\frac{\frac{d^2a}{dw^2}}{\frac{da}{dw}} \quad (76d)$$

Under positive linear transformation falling in line with Von Neumann-Morgenstern utility functional, the aversion does not depend on k . The changes in actuarial risk aversion with changing level of wealth is thus linked to the boundedness of the utility function. The decision of the insured is to choose the value k invested in risky insurance portfolio in order to maximize the function.

$$E[u(w)] = E[u(ky + \alpha)] = v(k) \quad (76e)$$

If $0 \leq k \leq \alpha$ where the policy is viable with rate of return δ then in the transformation above,

$$\alpha = \bar{\alpha}(1 + \delta) \Rightarrow E[u(w)] = E[u(ky + \bar{\alpha}(1 + \delta))] \quad (76f)$$

$$\frac{d}{dk} v(k) = E\left[\frac{d}{dw} u(w)\right] = E\left[y \frac{d}{dw} u(ky + \alpha)\right] = E\left[y \frac{d}{dw} u(w)\right] \quad (76g)$$

$$\frac{d^2}{dk^2} v(k) = E\left[\frac{d^2}{dw^2} u(w)\right] = E\left[y^2 \frac{d^2}{dw^2} u(w)\right] \quad (76h)$$

$$\frac{d^m}{dk^m} v(k) = E\left[\frac{d^m}{dw^m} u(w)\right] = E\left[y^m \frac{d^m}{dw^m} u(w)\right] \quad (76i)$$

Consequently, if the expected utility is maximized, then apparently, the insured will tend to be risk-averse to small risks in w provided that the utility functional $u(y)$ remains concave at w such that $E(u(y)) < U(E(y))$ (76j)

(iii) Again, substituting equation (75) in (71), we have $\Sigma^+ = \mu_y + \frac{\mu_2}{2} a(w, \mu_y)$ (77)

This is the estimated maximum premium the insured can pay. This equation means that $u(y)$ must have a continuous and bounded third order derivative for all risk neutral. The risk premium is proportional to the variance of its random loss. The variance thus seems to be a good measure of the degree of an insurable risk. This observation induces the application of mean-variance technique for evaluating risk behavior under uncertainty. Under the mean-variance model, it is assumed that individual risk behavior is a function of the mean and variance of the underlying risks. Furthermore, the validity of these models depends on the precision of the approximation in the aversion to risk co-efficient which appears accurate given that the risk is infinitesimally small. The risk premium corresponding to a large risk depend on the other moment of the distribution of the risk but not just on its mean and variance hence we examine the intensity of symmetry of Y about its mean in obtaining the risk premium. From equation (65), the degree of skewness could also affect the desirability of a risk and consequently two risks with the same mean and variance but the one with a distribution that is skewed to the right and the other with a distribution that is skewed to the left should not be expected to have the same risk premium.

- (v) Suppose the risk premium increases with the size of the risk and the variance is proportional to the square of this size for small risk. Consequently if $Y = \alpha\theta$ and $E(\theta) = 0$ the parameter α defines the magnitude of the risk and hence as $\alpha \rightarrow 0$, the risk tends infinitesimally small to zero. As the risk premium is a function of the size of the risk, the risk premium function $\pi(\alpha)$ is expected to increase in α and hence the functional form connecting the risk premium π to the magnitude α of the risk can be computed. Because the variance

$$Var(Y) = \alpha^2 Var(\theta) \tag{78}$$

We can see that

$$(\mu_Y - \Sigma^+) = \left[\frac{\mu_Y u''(w - \mu_Y)}{2u'(w - \mu_Y)} \right] = \left[\frac{\alpha^2 Var(\theta) u''(w - \mu_Y)}{2u'(w - \mu_Y)} \right] = \pi(\alpha) \tag{79}$$

The risk premium is approximately proportional to the square of the size of the risk. It is apparent that $\pi(\alpha)$ tends zero as α approaches zero

$$\frac{d\pi(\alpha)}{d\alpha} = \frac{\alpha Var(\theta) u''(w - \mu_Y)}{u'(w - \mu_Y)} \tag{80}$$

$$\lim_{\alpha \rightarrow 0} \left[\frac{\alpha Var(\theta) u''(w - \mu_Y)}{u'(w - \mu_Y)} \right] = 0 \tag{81}$$

$$\frac{d^2\pi(\alpha)}{d\alpha^2} = \frac{Var(\theta) u''(w - \mu_Y)}{u'(w - \mu_Y)} \tag{82}$$

Considering the marginal second derivative, a small zero-mean risk has no effect on the payoff of risk averse insurance agents.

6. Conclusion

The problem of evaluating proper aversion co-efficient in actuarial risk has received considerable attention with a considerable appeal to actuaries interested in applying numerical techniques and probability theory to solving actuarial risk aversion problems. The underwritten risk is the uncertainty resulting from an insurance scheme associated with random risks affecting underwriting results of insurance that could simultaneously impact on premium. In this paper, we have succinctly discussed utility function and its actuarial consequences on the construction and evolution of aversion co-efficient serving as a working tool in actuarial literature. This is based on the general reasoning of actuarial computations which may account for uncertainties and other risk criteria about underwriters and insured together with their utility preferences. For any reasonable actuarial risk aversion to be constructed, the utility functions should be modelled to fall in line with risk preferences. From our arguments, the theory of aversion can be applied either by the scheme holder who wishes to compute the maximum premium he can

affordably pay so as to secure full cover or by the underwriter to determine the minimum premium to charge the scheme holder. The findings on aversion apply to the evaluation of risk profile trajectories guiding the insurer to accept or reject volatile schemes which was generated under conventional deterministic frame work. The degree of risk exposure has a pervasive impact on the profitability level of insurance underwriting and could assume a major source of generating viral insolvency of financial losses. All risk mitigations are geared towards boosting the insurance profit, increasing the market share and ensuring that the financial security is required by both risk managers and insured to identify evaluate risk aversion of risk depending on the efficiency of operations and the competitiveness of the market. Risk aversion is evaluated as a consequence of interplay of key parameters such as the probability of deviation from the intended purpose for which it is modeled. It is believed in insurance practice, that the elimination of the effect of risk factor subsequently may offer optimal financial stability. Insurance companies are therefore encouraged to deploy potent risk management assessment tools which will mitigate the overall measures of insurance business risk. Risk aversion assessment dynamics involves defined sequence of algorithms which will ensure adequate impact on risk measurement comprising analysis of risk data, numerical computation of risk probabilities, identifying the intensity and amount of risk. The loss of premium income from insurance policy and a sharp reduction in risk-free profit evolves from random strategies of the insurance firms when the insured is wrongly underwritten and accepted.

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