DIU Journal of Business and Entrepreneurship Volume 14, No. 01, 2021
https://doi.org/10.36481/diujbe.v014i1.h5ewn619

Application of Stochastic Differential Equation in Insurance Portfolio Construction Involving Leverage Function and Elasticity of Debt and Equity

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Abstract: *Leverage effect specifies the functional relationship between stock returns and volatility. As stock price declines, volatility tends to rise. Thus, the variability in market prices of a company's stock has pervasive effect when measuring the level of leverage in the capital structure. The determination of portfolio value of an investor using the continuous time second order stochastic differential equation has major consequence on leverage function. Usually, structural stochastic value of leveraged firms treats company's portfolio as equity whose underlying instrument is the company's asset. In this paper, the objectives are to theoretically: (i) measure the value of insurance company's portfolio by second order stochastic differential equation, (ii) apply Ito's rule to obtain a value on its leverage function and, (iii) obtain the analytical correspondence between equity and volatility in a leveraged company through infinitesimal calculus. The stochastic second order differential equation of portfolio value under two arguments results in equilibrium position which provides the traded price of the derivative, furthermore the linear combination of first order derivative of volatilities with respect to equity and debt is vanishingly zero based on the underlying elasticity of stock volatilities. The resultant effect is that elasticity* Ξ *of debt and equity cancel out and* $-1 \le \Xi_{\zeta} \le 0 \le \Xi_{\chi} \le 1$

Keywords: Elasticity, Leverage function, Portfolio, Stochastic, Volatility

1. Introduction

Portfolio describes a bundle of financial instruments such as stocks and bonds constructed to mitigate the risk of poor performance of different combination of investments depending on investor's level of wealth and the amount of market risk a fund manager can tolerate. The risk tolerance affords the investor to cope with losses at the time the investment performance seems undesirable especially under randomness (Grundl et al. 2016; Henriques & Neves, 2019). Investors could initiate a theoretical portfolio where each profitable instrument is inclusive at a degree which varies in proportion to its market value. In other words, market value of a portfolio defines a

linear combination of stock prices and other derivative instruments.
Let, $\eta = k_1\eta_1 + k_2\eta_2 + k_3\eta_3 + ... + k_m\eta_m$ define some linear combination of stock and let $d = g_1 d_1 + g_2 d_2 + g_3 d_3 + ... + g_n d_n$ be the linear combination of derivatives d where k explains how many units are held for stock S while g shows how many units are held for the derivative d, consequently, the stock value . $Z = \eta + d$ and this is equal to

$$g_1 d_1 + g_2 d_2 + g_3 d_3 + ... + g_n d_n + k_1\eta_1 + k_2\eta_2 + k_3\eta_3 + ... + k_m\eta_m$$

$$V = \sum_{i=1}^{r} K_i \eta_i + \sum_{j=1}^{m} g_j d_j;$$

$$\Delta V = \sum_{i=1}^{r} K_i \Delta\eta_i + \sum_{j=1}^{s} g_j \Delta d_j$$

When $\Delta Z = \alpha Z \Delta\xi$, then the increment ΔZ can be estimated and equal to the default free interest earned in time interval Δt making the portfolio αZ or ΔZ riskless where $\alpha = constant$ and is the default free interest rate. The technique of formulating the required investment policy by fund managers by reason of the lowest of riskiness and largest gain within a defined time interval in the presence of variations of market conditions evolves out of the portfolio management conditions. An investment fund such as life fund is an investment scheme comprising lump sums of contributions gotten from plan members based on goal of investing in return bearing instruments. Investment funds are handled by qualified market professionals who invest the fund to subsequently generate capital gains on behalf of the contributing members in the scheme. We observe in pension sector, that contributors are billed service fees for the services rendered on their behalf by pension managers. However, investors who manage mutual funds contribute one off payment and hence are given permission to raise the volume of the investment in the scheme when they inject supplementary capital consequently, periodical addition to the mutual fund is allowed nevertheless removal from the scheme is prohibited until the scheme matures. The scheme's management is scheduled in managing the scheme's asset within the rules and conditions of the scheme till it matures so as to meet up the satisfactory investment adequacy of members. This paper will be markedly useful in the following settings: The solution to the second order differential equation can be adopted to obtain the tradeable value of security such that it can be traded on the floor of stock exchange. Consequently, the resulting equilibrium position can be adopted to obtain the traded price of an underlying derivative. When selecting investment instruments, it is instructive to consider a range of results under multiple settings. Insurance firms can use the stochastic construction to enhance their investment practices and increase profit margin. In particular the portfolio managers in financial services sector can use the stochastic construction framework to analyze portfolio trajectories & equity market structure, manage assets & liabilities and then optimize their portfolios.

2. Literature Review

Optimized portfolio selection problems are usually examined by adopting control theory such as Merton (Merton, 1969; Merton, 1971) where the problem was studied under the structure of continuous-time financial framework. The author was able to arrive at a closed-form solution where the investor is planning on which percentage of his wealth should be assigned between consumption and investment and where the utility preferences are obtained under constant relative risk-averse utility function. The authors (Ihedioha, Ajai & Ogungbenle, 2020) were motivated to apply logarithmic utility optimization of the investment strategy of an insurer using the modified constant elasticity of variance (M-CEV) model in investigating various optimal investment choice problems. Furthermore, (Ihedioha, Ogungbenle, & Auta 2020) examined optimal investment strategy for a fund manager with Ornstein-Uhlenbeck and constant elasticity of variance (CEV) models under correlating and non-correlating Brownian motion. These papers investigated the utility optimization of an insurer who attempts to optimize the expected satisfaction of final value of his asset portfolio on a finite time horizon such that he distributes his wealth among default-free asset and risky instruments. The analytical portfolio optimization theory initiated by Markowitz and adopted in a multi-period continuous-time domain by (Merton, 1969; Merton, 1971) assume insurers invest in risky assets. The time-wasting and bigger opportunity cost of trading securities is the rationale behind investors changing from direct investment in equities to mutual schemes. The adoption of continuous-time mechanisms to analyze investment portfolio is found in (Cetin, 2006; Hugonnier & Kaniel, 2010). The connecting domain of these papers is the massive usage of information availability to investors when building financial market portfolio. However, the following papers (Lakner,1995; Lakner, 1998; Feldman, 2007; Monoyios, 2010; Monoyios, 2007; Bjork et al., 2010 and Liu & Muhle-Karbe, 2013) examined continuous-time portfolio choice under partial information. It also examines utility optimization problem of the investor optimizing his expected utility of final wealth within the finite time domain $[0,T]$ by investing his wealth, $W(\xi)$ at any time $\xi \in [0,T]$ in a risky asset. The motivation is that the authors did not limit the condition that every financial investor should possibly adopt the return vector process $\mu(\xi)$ together with the Brownian motion $B(\xi)$ adopted in the computational modelling of the evolving risky asset prices. Thus, the drift vector, $\mu(\xi)$ and $B(\xi)$ are not real financial variables and consequently are indirectly observed to economic agents. The researches on optimal strategy of an investor under the constant elasticity of variance and the Ornstein-Uhlembeck models confirm the results which may occur given that the Brownian motions do not correlate. An insurer intends to optimize the expected satisfaction of final wealth where he is allowed to invest in a risk-free instrument and risky instruments. By adopting optimal control, the Hamilton-Jacobi-Bellman equation associated with the assistance of the maximum principle can be changed into a complex non-linear partial differential equation. In view of the tasking level of the solution, power transform with change of variable methodology to ease the PDE is used and arrive at an explicit solution of the power utility model. Based on the techniques of (Black & Scholes, 1973; Gao, 2009), the non-linear second-order partial differential equation are usually transformed into a linear form by adopting elimination of dependency on the variables W (net worth) and π (price of the risky asset). Under the continuous time structure of stochastic domain, the state variable in the stochastic differential equation is wealth or net worth while the controls are the stocks put at every time in diverse assets. Given the asset allocation chosen at a time, the determinants of the change in wealth are the stochastic returns to assets and the interest rate on the risk-free asset. In this paper, it is intended to obtain the stochastic second order differential equation which models the time dependent value $Z(.)$ of a security of an investor. In (Chadha & Sharma, 2015; Korkmaz, 2016; Lenka, 2017), while studying capital structure and debt ratios built a link between key indicators of business competitiveness and performance. In particular, (Lenka, 2017) proved that corporate leverage varies across industries. In a study conducted in (Sinha, 2013) to study profit elasticity, the author measured the degree of operating leverage along conditions for the existence and non-existence of operating leverage effect using non-linear profit function.

3. Mathematical Preliminaries: Stochastic Processes

A stochastic process describes a collection of $(\mathrm{Y_s})_{\mathrm{s \in B}}$ of random variables on probability space $(\Omega,\mathrm{F},\mathrm{P})$. The index set of the process is B , $\mathrm{Y{:}B \times \Omega \rightarrow R}$

When $\mathrm{B} = \{0,1,2,3,..\}$, then $\mathrm{Y}(\xi)$ is a discrete process but when B is an interval such that $\mathrm{B} \in [o,\infty]$, then $\mathrm{Y}(\xi)$ is a continuous process.

A stochastic process η_ξ is described as Ito's process if $d\eta_\xi = \mu_\xi d\xi + \sigma_\xi dW_\xi$, μ_ξ is the drift, σ_ξ is the volatility parameter and dW_ξ is the increment in Brownian motion. In stochastic calculus, the following properties are observed

$$\left.\begin{array}{rcll} dW_\xi dW_\xi & = & d\xi & (i) \\ dW_\xi d\xi & = & 0 & (ii) \\ d\xi dW_\xi & = & 0 & (iii) \\ d\xi d\xi & = & 0 & (iv) \end{array}\right\}$$

3.1 Ito's Lema

Suppose stochastic process η_ξ is an Ito process and $G(\eta,\xi)$ is twice continuously differentiable function on η and ξ, then

$$dG = \left(G_\xi + \mu_\xi G_\eta + \frac{1}{2}\sigma_\xi^2 G_{\eta\eta} \right) d\xi + \sigma_\xi dW_\xi G_\eta$$

This is the chain rule applicable in stochastics of bivariate process. However, when we have a multivariate process, it is given as

$$dG = \left(G_\xi + \sum_{k=1}^{m} \mu_k G_k + \frac{1}{2}\sum_{k=1}^{m}\sum_{k=1}^{m} \sigma_r \sigma_r \delta_{kr} G_{kr} \right) d\xi + \sum_{k=1}^{m} \sigma_k G_k dW_k\left(\xi\right)$$

$$G_k = \frac{\partial G}{\partial \eta_k}; G_{kr} = \frac{\partial^2 G}{\partial \eta_k \partial \eta_r}; \delta_{kr} d\xi = dW_k\left(\xi\right) dW_r\left(\xi\right)$$

Obtaining a closed form solution for differential coefficients characterized by multivariate stochastic process as mentioned above is very difficult. The value of derivatives is solved by Monte Carlo simulation. To simulate the system characterized by multiple correlated Brownian motion, the Cholesky decomposition is applicable which is the splitting of a Hermitian and positive definite matrix B into the product of lower triangular matrix L, that is real and positive diagonal entries and its conjugate transpose such that $B = LL^T$. Using Cholesky-Crout and Cholesky-Banachiewicz decomposition,

$$B = \begin{pmatrix} L_{11} & 0 & . & . & 0 \\ L_{21} & L_{22} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ L_{K1} & L_{K2} & . & . & L_{kk} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & . & . & L_{K1} \\ 0 & L_{22} & . & . & L_{K2} \\ . & . & . & . & \\ . & . & . & . & \\ 0 & 0 & . & . & L_{kk} \end{pmatrix}$$

$$L_{k,k} = \left(A_{k,k} - \sum_{p=1}^{k-1} L^2{}_{k,p} \right)^{\frac{1}{2}}, \; L_{k,r} = \frac{A_{k,r} - \sum_{p=1}^{r-1} L_{k,p} L_{r,p}}{L_{rr}}; k > r,$$

4. Construction of the Value of Portfolio

The method of construction looks only at a combination of stochastic and infinitesimal calculus starting with second order differential equation of portfolio value in two arguments. The main objectives of the study are to theoretically (1) measure the value of a company's portfolio by second order stochastic differential equation, (2) apply Ito's rule to obtain the value of its leverage function (3) determine the analytical relationship between equity and volatility in a leveraged company through infinitesimal calculus. The study combines stochastic and analytical approach.

4.1 Theorem 1

Let security $Z(\eta,\xi), 0<\xi<\mathrm{T}$ be purchased and the underlying stock η is sold with quantity δ to build a portfolio P at time ξ, if $Z(\eta,\xi)=\mathrm{P}+\delta\eta$ (1)

then $\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}+2\frac{\partial Z(\eta,\xi)}{\partial\xi}-2\mathrm{r}\frac{\partial Z(\eta,\xi)}{\partial\eta}\eta+2\mathrm{r}\mathrm{Z}(\eta,\xi)=0$ (2)

is the second order differential equation governing the system subject to the condition

$$\delta=\frac{\partial Z(\eta,\xi)}{\partial\eta} \quad (3)$$

4.1.1 Proof

Differentiating both sides, we have

$\mathrm{dP}=\mathrm{dZ}(\eta,\xi)-\delta\mathrm{d}\eta$, where δ is constant during time step, therefore adopting Ito's Lema, hence

$$\mathrm{dZ}(\eta,\xi)=\frac{\partial Z(\eta,\xi)}{\partial\xi}\mathrm{d}\xi+\frac{\partial Z(\eta,\xi)}{\partial\eta}\mathrm{d}\eta+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}\mathrm{d}\xi \quad (4)$$

so that $\mathrm{dP}=\frac{\partial Z(\eta,\xi)}{\partial\xi}\mathrm{d}\xi+\frac{\partial Z(\eta,\xi)}{\partial\eta}\mathrm{d}\eta+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}\mathrm{d}\xi-\delta\mathrm{d}\eta$ (5)

$$\mathrm{dP}=\frac{\partial Z(\eta,\xi)}{\partial\xi}\mathrm{d}\xi+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}\mathrm{d}\xi+\frac{\partial Z(\eta,\xi)}{\partial\eta}\mathrm{d}\eta-\delta\mathrm{d}\eta \quad (6)$$

$$\mathrm{dP}=\frac{\partial Z(\eta,\xi)}{\partial\xi}\mathrm{d}\xi+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}\mathrm{d}\xi+\left(\frac{\partial Z(\eta,\xi)}{\partial\eta}-\delta\right)\mathrm{d}\eta \quad (7)$$

The increments $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are deterministic and random while the term $\left(\frac{\partial Z(\eta,\xi)}{\partial\eta}-\delta\right)$ is the element of risk function. In order to mitigate the randomness, the condition is for the random term to vanish such that $\frac{\partial Z(\eta,\xi)}{\partial\eta}-\delta=0\Rightarrow\frac{\partial Z(\eta,\xi)}{\partial\eta}=\delta$ (8)

and substituting in the portfolio construction, we have

$$\mathrm{dP}=\frac{\partial Z(\eta,\xi)}{\partial\xi}\mathrm{d}\xi+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}\mathrm{d}\xi+0\times\mathrm{d}\eta \quad (9)$$

$$\mathrm{dP}=\frac{\partial Z(\eta,\xi)}{\partial\xi}\mathrm{dt}+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}\mathrm{d}\xi \quad (10)$$

Under the no-arbitrage condition,

$$\frac{\partial Z(\eta,\xi)}{\partial\xi}=\mathrm{rP}\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}=0 \quad (11)$$

so that $\frac{dP}{d\xi} = rP = r\left[\delta\eta - Z(\eta,\xi)\right] \Rightarrow dP = r\left[\delta\eta - Z(\eta,\xi)\right]d\xi$ (12)

Substituting, into the no-arbitrage condition above, we have

$$rPd\xi = \frac{\partial Z(\eta,\xi)}{\partial\xi}d\xi + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2}d\xi \Rightarrow rP = \frac{\partial Z(\eta,\xi)}{\partial\xi} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2} \quad (13)$$

since $d\xi$ is common.

since $P + \delta\eta = Z(\eta,\xi)$ we have (14)

$$\frac{\partial Z(\eta,\xi)}{\partial\xi} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2} = r\left[\delta\eta - Z(\eta,\xi)\right] \quad (15)$$

$$\frac{\partial Z(\eta,\xi)}{\partial\xi} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2} = r\delta\,\eta - rZ(\eta,\xi) \quad (16)$$

recall, that $\delta = \frac{\partial Z(\eta,\xi)}{\partial\eta}$

$$\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi)}{\partial\eta^2} + 2\frac{\partial Z(\eta,\xi)}{\partial\xi} - 2r\frac{\partial Z(\eta,\xi)}{\partial\eta}\eta + 2rZ(\eta,\xi) = 0 \quad (17)$$

The solution $Z(\eta,\xi)$ to the second order differential equation is regarded as a tradeable value of security because it can be traded on floor of stock exchange. It results in equilibrium position which provides the traded price of the derivative. The value of derivative functionally depends on the volatility σ at time t and the underlying price η so that $Z = Z(\xi,\eta,\sigma)$, price and volatility are defined by stochastic processes

$$d\eta(\xi) = \eta(\xi)rd\xi + \sigma(\xi)\eta(\xi)dW_1(\xi) \quad (18)$$

$$d\sigma(\xi) = \alpha(\xi,\eta,\sigma)d\xi + \theta(\xi,\eta,\sigma)dW_2(\xi) \quad (19)$$

$$d\eta(\xi).d\sigma(\xi) = \left[\eta(\xi)rd\xi + \sigma(\xi)\eta(\xi)dW_1(\xi)\right].\left[\alpha(\xi,\eta,\sigma)d\xi + \theta(\xi,\eta,\sigma)dW_2(\xi)\right] \quad (20)$$

$$d\eta(\xi).d\sigma(\xi) = \left[\eta(\xi)rd\xi + \sigma(\xi)\eta(\xi)dW_1(\xi)\right].\left[\alpha(\xi,\eta,\sigma)d\xi + \theta(\xi,\eta,\sigma)dW_2(\xi)\right] \quad (21)$$

$$\begin{aligned} d\eta(\xi).d\sigma(\xi) &= \alpha(\xi,\eta,\sigma)d\xi\eta(\xi)rd\xi + \theta(\xi,\eta,\sigma)dW_2(\xi)\eta(\xi)rd\xi + \\ &\alpha(\xi,\eta,\sigma)d\xi\sigma(\xi)\eta(\xi)dW_1(\xi) + \\ &\theta(\xi,\eta,\sigma)dW_2(\xi)\sigma(\xi)\eta(\xi)dW_1(\xi) \end{aligned} \quad (22)$$

$$d\eta(\xi).d\sigma(\xi) = \theta(\xi,\eta,\sigma)\sigma(\xi)\eta(\xi)dW_2(\xi)dW_1(\xi) \quad (23)$$

$$d\eta(\xi).d\sigma(\xi) = \theta(\xi,\eta,\sigma)\sigma(\xi)\eta(\xi)d\xi = \lambda(\xi)d\xi \quad (24)$$

$$E\left(d\eta(\xi).d\sigma(\xi)\right)=E\left(\lambda(\xi)d\xi\right)=\lambda(\xi)d\xi \quad (25)$$

Where $\lambda(\xi)=\theta(\xi,\eta,\sigma)\sigma(\xi)\eta(\xi)$ (26)

5. Materials and Methods

5.1 Stochastic Calculation of Leverage Function

Because of the expectation defined above, a very related application is the zero mean return defined by stochastic differential equation $d\eta(\xi)=\sigma(\xi)dW(\xi)$. A basic assumption in stochastic calculus is that $\sigma(\xi)$ and $dW(\xi)$ are independent random variables. By the property of Wiener process that $E\left(W(\xi)\right)=0, E\left(d(\xi)^2\right)=0$ (27)

$$E\left(d\eta(\xi)\right)=E\left(\sigma(\xi)dW(\xi)\right)=E\left(\sigma(\xi)\right)E\left(dW(\xi)\right)=E\left(\sigma(\xi)\right)x0=0 \quad (28)$$

$$E\left(d\eta^2(\xi)\right)=E\left(\sigma^2(\xi)dW^2(\xi)\right)\ =E\left(\sigma^2(\xi)d\left(W(\xi)W(\xi)\right)\right)=$$

$$E\left(\sigma^2(\xi)\right)E\left[W(\xi)dW(\xi)+W(\xi)dW(\xi)+dW(\xi)dW(\xi)\right] \quad (29)$$

$$=E\left(\sigma^2(\xi)\right)E\left[2W(\xi)dW(\xi)+dW(\xi)dW(\xi)\right]=$$

$$E\left(\sigma^2(\xi)\right)E\left[2W(\xi)dW(\xi)+d\xi\right]=E\left(\sigma^2(\xi)\right)E(d\xi) \quad (30)$$

$$E\left(d\eta^2(\xi)\right)=E\left(\sigma^2(\xi)\right)d\xi \quad (31)$$

$$VAR\left(d\eta(\xi)\right)=E\left(d\eta^2(\xi)\right)-E\left(d\eta(\xi)\right)=E\left(\sigma^2(\xi)\right)E(dt)-0=E\left(\sigma^2(\xi)\right)d\xi \quad (32)$$

$$E\left(d\eta^3(\xi)\right)=E\left[\sigma^3(\xi)dW^3(\xi)\right]=E\left[\sigma^3(\xi)d\left(W(\xi)\ W^2(\xi)\right)\right]=$$

$$=E\left(\sigma^2(\xi)\right)E\left[W(\xi)d\left(W^2(\xi)\right)+W^2(\xi)dW(\xi)+d\left(W^2(\xi)\right)dW(\xi)\right] \quad (33)$$

$$=E\left(\sigma^3(\xi)\right)E\begin{bmatrix}W(\xi)\left\{2W(\xi)dW(\xi)+d\xi\right\}+W^2(\xi)dW(\xi)\\ +dW(\xi)\left\{2W(\xi)dW(\xi)+d\xi\right\}\end{bmatrix} \quad (34)$$

$$=E\left(\sigma^3(\xi)\right)E\begin{bmatrix}\left\{2W^2(\xi)dW(\xi)+W(\xi)d\xi\right\}+W^2(\xi)dW(\xi)\\ +\left\{2W(\xi)\left(dW(\xi)\right)^2+dW(\xi)d\xi\right\}\end{bmatrix} \quad (35)$$

$$=E\left(\sigma^3(\xi)\right)E\begin{bmatrix}2W^2(\xi)dW(\xi)+W(\xi)d\xi+W^2(\xi)dW(\xi)\\ +2W(\xi)\left(dW(\xi)\right)^2+dW(\xi)d\xi\end{bmatrix} \quad (36)$$

$$=E\left(\sigma^3(\xi)\right)E\left[2W^2(\xi)dW(\xi)+W(\xi)d\xi+W^2(\xi)dW(\xi)+2W(\xi)d\xi\right] \quad (37)$$

$$= E\left(\sigma^3(\xi)\right)E\left[3W^2(\xi)dW(\xi)+3W(\xi)d\xi\right]=$$
$$E\left(\sigma^3(\xi)\right)\left\{3E\left[W^2(\xi)\right]E\left[dW(\xi)\right]+3E\left[W(\xi)\right]E\left[d\xi\right]\right\} \tag{38}$$

$$= E\left(\sigma^3(\xi)\right)\{3\xi x0+3x0xd\xi\}=0 \tag{39}$$

$$E\left(d\eta^4(\xi)\right)=E\left[\sigma^4(\xi)dW^4(\xi)\right]=E\left[\sigma^4(\xi)d\left(W(\xi)\ W^3(\xi)\right)\right] \tag{40}$$

$$= E\left(\sigma^4(\xi)\right)E\left[W(\xi)d\left(W^3(\xi)\right)+W^3(\xi)dW(\xi)+d\left(W^3(\xi)\right)dW(\xi)\right] \tag{41}$$

$$= E\left(\sigma^4(\xi)\right)E\begin{bmatrix} W(\xi)\left\{\left[3W^2(\xi)dW(\xi)+3W(\xi)d\xi\right]\right\}+W^3(\xi)dW(\xi)+ \\ dW(\xi)\left\{\left[3W^2(\xi)dW(\xi)+3W(\xi)d\xi\right]\right\} \end{bmatrix} \tag{42}$$

$$= E\left(\sigma^4(\xi)\right)E\begin{bmatrix} \left\{3W^3(\xi)dW(\xi)+3W^2(\xi)d\xi\right\}+W^3(\xi)dW(\xi)+ \\ \left\{3W^2(\xi)\left(dW(\xi)\right)^2+3W(\xi)dW(\xi)d\xi\right\} \end{bmatrix} \tag{43}$$

$$= E\left(\sigma^4(\xi)\right)E\left[3W^3(\xi)dW(\xi)+3W^2(\xi)d\xi+W^3(\xi)dW(\xi)+3W^2(\xi)d\xi\right] \tag{44}$$

$$= E\left(\sigma^4(\xi)\right)E\left[4W^3(\xi)dW(\xi)+6W^2(\xi)d\xi\right] \tag{45}$$

$$= 6E\left[\left(\sigma^4(\xi)\right)W^2(\xi)d\xi\right]=6\left(\sigma^4(\xi)\right)\xi d\xi,\ E\left[W^2(\xi)\right]=\xi \tag{46}$$

$$VAR\left(d\eta^2(\xi)\right)=E\left(d\eta^4(\xi)\right)-E\left(d\eta(\xi)\right)=6E\left[\left(\sigma^4(\xi)\right)W^2(\xi)d\xi\right]-$$
$$\left[E\left(\sigma^2(\xi)\right)d\xi\right]^2 \tag{47}$$

$$VAR\left(d\eta^2(\xi)\right)=6E\left(\sigma^4(\xi)\right)\xi d\xi-\left[E\left(\sigma^2(\xi)\right)d\xi\right]^2 \tag{48}$$

This describes a useful property of Wiener process

$$E\left(dW^2(\xi+\zeta)\right)=d\xi,\zeta>0 \tag{49}$$

$$E\left(d\eta(\xi)d\eta^2(\xi+\zeta)\right)=E\left(d\eta(\xi)\right)E\left(d\eta^2(\xi+\zeta)\right)=$$
$$E\left(\sigma(\xi)dW(\xi)\right)E\left(\sigma(\zeta+\xi)dW^2(\xi+\zeta)\right) \tag{50}$$

$$= \begin{matrix} E\left(\sigma(\xi)dW(\xi)\right)E\left(\sigma(\zeta+\xi)d\xi\right)=E\left(\sigma(\xi)dW(\xi)\right)E\left(\sigma(\zeta+\xi)\right)E\left(d\xi\right)= \\ E\left(\sigma(\xi)dW(\xi)\right)E\left(\sigma(\zeta+\xi)\right)d\xi \end{matrix} \tag{51}$$

If $\zeta=0,\ E\left(d\eta(\xi)d\eta^2(\xi+\zeta)\right)=E\left(d\eta(\xi)d\eta^2(\xi)\right)=E\left(d\eta^3(\xi)\right)=0$ (52)

The non-trivial association between price changes at different time ξ is the Leverage function defined by

$$L_Z=\frac{E\left(d\eta^2(\zeta+\xi)d\eta(\xi)\right)}{E\left(d\eta^2(\xi)\right)^2}=\frac{E\left(\sigma(\zeta+\xi)dW^2(\xi+\zeta)\right)}{\left[E\left(\sigma^2(\xi)\right)d\xi\right]^2}=\frac{E\left(\sigma(\zeta+\xi)\right)E\left[dW^2(\xi+\zeta)\right]}{\left[E\left(\sigma^2(\xi)\right)dZ\right]^2}. \tag{53}$$

$$L_Z = \frac{E\left(\sigma\left(\zeta + \xi\right)\right)d\xi}{\left[E\left(\sigma^2\left(\xi\right)\right)d\xi\right]^2} \tag{54}$$

$$L_Z = \frac{E\left(\sigma\left(\zeta + \xi\right)^2 d\eta\left(\xi\right)\right)}{E\left(\sigma\left(\xi\right)^2\right)^2} = \frac{E\left(\sigma^2\left(\xi + \zeta\right)\right)E\left(d\eta\left(\xi\right)\right)}{E\left(\sigma^2\left(\xi\right)\right)^2} \tag{55}$$

Thus the effect of financial leverage on equity return volatility in equilibrium dynamics of an economy characterized by debt and equity claims has been obtained and may have effect on market portfolio or on small company with market risk. It is observed that leverage effect function drives infinitesimally small changes in equity return volatility at the financial market level, however there exists considerable variation at the level of each company when interest rate remains constant. Furthermore, if there is a significant variation in interest rate, then there will be considerable changes in equity return volatility at market level and portfolio level so that leverage effect function will have small effect on the dynamics of equity return volatility at the market level and consequently contributing more to the dynamics of equity return volatility in respect of a small company.

In three dimensional arguments, we can construct a portfolio q which comprises of security having value $Z\left(\xi,\eta,\sigma\right)$ with quantity δ of the asset and another security with value $\overline{Z}\left(\xi,\eta,\sigma\right)$ with quantity $\overline{\delta}$ of the corresponding asset, then

$$q + \delta\eta + \overline{\delta}\overline{Z}\left(\eta,\xi,\sigma\right) = Z\left(\eta,\xi,\sigma\right),\ q = Z\left(\eta,\xi,\sigma\right) - \delta\eta - \overline{\delta}\overline{Z}\left(\eta,\xi,\sigma\right) \tag{56}$$

differentiating both sides, $dq + \delta d\eta + \overline{\delta}d\overline{Z}\left(\eta,\xi,\sigma\right) = dZ\left(\eta,\xi,\sigma\right)$ (57)

If $Z = Z\left(\eta,\xi,\sigma\right)$, then

$$dq = \frac{\partial Z\left(\eta,\xi,\sigma\right)}{\partial\xi}d\xi + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z\left(\eta,\xi,\sigma\right)}{\partial\eta^2}d\xi + \frac{1}{2}\theta^2\frac{\partial^2 Z\left(\eta,\xi,\sigma\right)}{\partial\sigma^2}d\xi + \lambda\sigma\eta\theta\frac{\partial^2 Z\left(\eta,\xi,\sigma\right)}{\partial\eta^2}d\xi + \frac{\partial Z\left(\eta,\xi,\sigma\right)}{\partial\eta}d\eta + \frac{\partial Z\left(\eta,\xi,\sigma\right)}{\partial\sigma}d\sigma \tag{58}$$

$$dq = \frac{\partial Z\left(\eta,\xi,\sigma\right)}{\partial\xi}d\xi + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z\left(\eta,\xi,\sigma\right)}{\partial\eta^2}d\xi + \frac{1}{2}\theta^2\frac{\partial^2 Z\left(\eta,\xi,\sigma\right)}{\partial\sigma^2}d\xi$$

$$+\lambda\sigma\eta\theta\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}d\xi+\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}d\eta+\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}d\sigma-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\xi}d\xi$$
$$-\bar{\delta}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2\bar{Z}(\eta,\xi,\sigma)}{\partial\eta^2}d\xi-\bar{\delta}\frac{1}{2}\theta^2\frac{\partial^2\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma^2}d\xi-\bar{\delta}\lambda\sigma\eta\theta\frac{\partial^2\bar{Z}(\eta,\xi,\sigma)}{\partial\eta^2}d\xi-\delta d\eta$$
$$-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta}d\eta-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma}d\sigma \tag{59}$$

$$dq=\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}d\xi+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}d\xi+\frac{1}{2}\theta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\sigma^2}d\xi+\lambda\sigma\eta\theta\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}d\xi$$
$$-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\xi}d\xi-\bar{\delta}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2\bar{Z}(\eta,\xi,\sigma)}{\partial\eta^2}d\xi-\bar{\delta}\frac{1}{2}\theta^2\frac{\partial^2\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma^2}d\xi$$
$$-\bar{\delta}\lambda\sigma\eta\theta\frac{\partial^2\bar{Z}(\eta,\xi,\sigma)}{\partial\eta^2}d\xi+\left(\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}-\delta\;-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta}\right)d\eta$$
$$+\left(\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma}\right)d\sigma \tag{60}$$

The increment $d\xi, d\eta, d\sigma$ are deterministic and random while the terms $\left(\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}-\delta\;-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta}\right)d\eta; and \left(\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma}\right)d\sigma$ are the elements of risk function. In order to mitigate the degree of uncertainty and randomness, the following conditions are that the random terms must vanish such that

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}-\delta\;-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta}=0 \tag{61}$$

and $$\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma}=0 \tag{62}$$

so that $$\delta=\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}\;-\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta} \tag{63}$$

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}=\bar{\delta}\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta}\Rightarrow\delta=0 \tag{64}$$

$$\bar{\delta}=\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\eta}}{\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\eta}}=\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}}{\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\sigma}}=\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}}{\frac{\partial\bar{Z}(\eta,\xi,\sigma)}{\partial\xi}}=\frac{\partial Z(\eta,\xi,\sigma)}{\partial\bar{Z}(\eta,\xi,\sigma)} \tag{65}$$

$$\delta = \frac{\partial Z(\eta,\xi,\sigma)}{\partial \eta} - \frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial \sigma}}{\frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \sigma}} \frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \eta} = \frac{\partial Z(\eta,\xi,\sigma)}{\partial \eta} - \frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi}}{\frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \xi}} \frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \eta} \quad (66)$$

$$\delta = \frac{\partial Z(\eta,\xi,\sigma)}{\partial \eta} - \frac{\partial Z(\eta,\xi,\sigma)}{\partial \overline{Z}(\eta,\xi,\sigma)} \frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \eta} \quad (67)$$

Substituting the conditions of randomness into the portfolio construction, we have

$$\begin{aligned} dq = {} & \frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi} d\xi + \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} d\xi + \frac{1}{2}\theta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \sigma^2} d\xi + \lambda\sigma\eta\theta \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} d\xi \\ & -\bar{\delta} \frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \xi} d\xi - \bar{\delta}\frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \eta^2} d\xi - \bar{\delta}\frac{1}{2}\theta^2 \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \sigma^2} d\xi \\ & -\bar{\delta}\lambda\sigma\eta\theta \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \eta^2} d\xi + (0)d\eta + (0)d\sigma \end{aligned} \quad (68)$$

$$\begin{aligned} dq = {} & \frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi} d\xi + \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} d\xi + \frac{1}{2}\theta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \sigma^2} d\xi + \lambda\sigma\eta\theta \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} d\xi \\ & -\bar{\delta} \frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \xi} d\xi - \bar{\delta}\frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \eta^2} d\xi - \bar{\delta}\frac{1}{2}\theta^2 \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \sigma^2} d\xi \\ & -\bar{\delta}\lambda\sigma\eta\theta \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \eta^2} d\xi \end{aligned} \quad (69)$$

Again, $\frac{dq}{d\xi} = rq = r\left[Z(\eta,\xi,\sigma) - \delta\eta - \bar{\delta}\overline{Z}(\eta,\xi,\sigma)\right] \Rightarrow$

$$dq = r\left[Z(\eta,\xi,\sigma) - \delta\eta - \bar{\delta}\overline{Z}(\eta,\xi,\sigma)\right] d\xi \quad (70)$$

$$\begin{aligned} & \frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi} d\xi + \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} d\xi + \frac{1}{2}\theta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \sigma^2} d\xi + \lambda\sigma\eta\theta \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} d\xi \\ & -\bar{\delta} \frac{\partial \overline{Z}(\eta,\xi,\sigma)}{\partial \xi} d\xi - \bar{\delta}\frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \eta^2} d\xi - \bar{\delta}\frac{1}{2}\theta^2 \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \sigma^2} d\xi \\ & -\bar{\delta}\lambda\sigma\eta\theta \frac{\partial^2 \overline{Z}(\eta,\xi,\sigma)}{\partial \eta^2} d\xi = r\left[Z(\eta,\xi,\sigma) - \delta\eta - \bar{\delta}\overline{Z}(\eta,\xi,\sigma)\right] d\xi \end{aligned} \quad (71)$$

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi} + \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2} + \frac{1}{2}\theta^2 \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \sigma^2} + \lambda\sigma\eta\theta \frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2}$$

$$-\bar{\delta}\frac{\partial \bar{Z}(\eta,\xi,\sigma)}{\partial \xi}-\bar{\delta}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \eta^2}-\bar{\delta}\frac{1}{2}\theta^2\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \sigma^2}$$

$$-\bar{\delta}\lambda\sigma\eta\theta\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \eta^2}=\mathrm{r}\left[Z(\eta,\xi,\sigma)-\delta\eta-\bar{\delta}\bar{Z}(\eta,\xi,\sigma)\right] \tag{72}$$

As $d\xi$ cancels out

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi}+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2}+\frac{1}{2}\theta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \sigma^2}+\lambda\sigma\eta\theta\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2}$$
$$-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\frac{\partial \bar{Z}(\eta,\xi,\sigma)}{\partial \xi}-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \eta^2}$$
$$-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\frac{1}{2}\theta^2\frac{\partial^2 \bar{Z}(\eta,\mathrm{V},\sigma)}{\partial \sigma^2}-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\lambda\sigma\eta\theta\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \eta^2}$$
$$=\mathrm{rZ}(\eta,\mathrm{V},\sigma)-\mathrm{r}\delta\,\eta-\mathrm{r}\frac{\partial Z(\eta,\mathrm{V},\sigma)}{\partial \bar{Z}(\eta,\mathrm{V},\sigma)}\bar{Z}(\eta,\xi,\sigma) \tag{73}$$

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial \xi}+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial \eta^2}+\frac{1}{2}\theta^2\frac{\partial^2 Z(\eta,\mathrm{V},\sigma)}{\partial \sigma^2}+\lambda\sigma\eta\theta\frac{\partial^2 Z(\eta,\mathrm{V},\sigma)}{\partial \eta^2}$$
$$-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\frac{\partial \bar{Z}(\eta,\mathrm{V},\sigma)}{\partial \xi}-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \eta^2}$$
$$-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\frac{1}{2}\theta^2\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \sigma^2}-\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)}\lambda\sigma\eta\theta\frac{\partial^2 \bar{Z}(\eta,\xi,\sigma)}{\partial \eta^2}$$
$$=\mathrm{rZ}(\eta,\xi,\sigma)-\mathrm{r}\frac{\partial Z(\eta,\mathrm{V},\sigma)}{\partial \bar{Z}(\eta,\mathrm{V},\sigma)}\bar{Z}(\eta,\xi,\sigma) \tag{74}$$

This is the differential equation for the value of the portfolio considering three arguments. This represents the non-equilibrium condition which provides the traded price of the derivative. The value of portfolio is dependent on the volatility σ, time t and the underlying price η, so that $Z=Z(\xi,\eta,\sigma)$, price and volatility are defined by stochastic processes. Recall from above that

$$\bar{\delta}=\frac{\dfrac{\partial Z(\eta,\xi,\sigma)}{\partial \eta}}{\dfrac{\partial \bar{Z}(\eta,\xi,\sigma)}{\partial \eta}}=\frac{\dfrac{\partial Z(\eta,\xi,\sigma)}{\partial \sigma}}{\dfrac{\partial \bar{Z}(\eta,\xi,\sigma)}{\partial \sigma}}=\frac{\dfrac{\partial Z(\eta,\xi,\sigma)}{\partial \xi}}{\dfrac{\partial \bar{Z}(\eta,\xi,\sigma)}{\partial \xi}}=\frac{\partial Z(\eta,\xi,\sigma)}{\partial \bar{Z}(\eta,\xi,\sigma)} \tag{75}$$

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}+\frac{1}{2}\theta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\sigma^2}+\lambda\sigma\eta\theta\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}$$

$$-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma}}\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma}}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2\overline{Z}(\eta,\xi,\sigma)}{\partial\eta^2}$$

$$-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma}}\frac{1}{2}\theta^2\frac{\partial^2\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma^2}-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma}}\lambda\sigma\eta\theta\frac{\partial^2\overline{Z}(\eta,\xi,\sigma)}{\partial\eta^2}$$

$$=\mathrm{r}\left[Z(\eta,\xi,\sigma)+\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\sigma}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma}}\overline{Z}(\eta,\xi,\sigma)\right] \qquad (76)$$

$$\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}+\frac{1}{2}\theta^2\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\sigma^2}+\lambda\sigma\eta\theta\frac{\partial^2 Z(\eta,\xi,\sigma)}{\partial\eta^2}$$

$$-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}}\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}}\frac{1}{2}\sigma^2\eta^2\frac{\partial^2\overline{Z}(\eta,\xi,\sigma)}{\partial\eta^2}$$

$$-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}}\frac{1}{2}\theta^2\frac{\partial^2\overline{Z}(\eta,\xi,\sigma)}{\partial\sigma^2}-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}}\lambda\sigma\eta\theta\frac{\partial^2\overline{Z}(\eta,\xi,\sigma)}{\partial\eta^2}$$

$$=\mathrm{r}\left[Z(\eta,\xi,\sigma)-\frac{\frac{\partial Z(\eta,\xi,\sigma)}{\partial\xi}}{\frac{\partial\overline{Z}(\eta,\xi,\sigma)}{\partial\xi}}\overline{Z}(\eta,\xi,\sigma)\right] \qquad (77)$$

Thus equations (76) and (77) are equivalent forms at points σ and ξ respectively of equation (74)

6. Application of Random Change in the Value of Assets using Infinitesimal Calculus

When acting on issues relating to financial judgement, capital structure takes higher priority in order of importance. Since this relates to Equity-Debt framework of a firm, the resultant risk-return of the stakeholders is of prime significance for insurance managers. Where the borrowed capital exceeds the owner's capital, then the *shareholder's expected returns will be proportionately increased*. Consequently, the insurance firm is exposed to financial risk. Nevertheless, if the fraction of the equity capital exceeds the fraction of the borrowed capital, the risk-return structure of the stakeholders tends to be very small. Consequently, this underscores the rationale behind adopting stochastic capital structure under which risk and return to stakeholders are matched. This technique on capital structure assists insurance firm to take efficient decisions on long-term and short-term strategies. Moreover, in the administration of insurance schemes where time horizon of many years constitutes terms and conditions, the short-term horizon within the framework of solvency may encapsulate investment decisions which could be very risk averse. However, in a larger perspective, it could result in a too cautious investment policy decisions at the expense of high profit margins that evolves in order of insurance business priority only when considering long-term horizons. Consequently, the adoption of leverage in respect of its behaviour which is usually measured by $\frac{Earnings\ before\ interest\ and\ tax}{Earnings\ before\ tax\ but\ after\ interest}$ could functionally assist to measure the financial structure in a judicious manner. A company may fund its activities using common stocks along with retained earnings or debt or a linear combination of these instruments. Therefore, the capital structure describing debt to equity ratio is indicator of how risky the firm is. Thus, the capital structure has significant influence on: the financial risk exposure of the debt holders, the expected profitability of the company and the cost of capital. In (Choi & Richardson, 2016; Maiti & Balakrishnan, 2020), leverage which describes the value of debt that an insurer uses to finance its assets is construed to mean the potentiality of the company to adopt long term capital having fixed costs to increase earnings per share. If χ is the default free loan value (debt), Z specifies the underlying asset value, ζ is the equity value, M is the number of shares and η is the market price per share, $Z = \zeta + \chi = M\eta + \chi$ (78)

δZ is the random change in asset value

$$\frac{\delta\eta}{\eta} = \frac{\delta\zeta}{\zeta} = \frac{\delta Z}{\zeta} = \frac{\delta Z}{\zeta} \times \left(\frac{Z}{Z}\right) = \frac{\delta Z}{Z}\left(\frac{Z}{\zeta}\right) = \frac{\delta Z}{Z}\left(\frac{\zeta + \chi}{\zeta}\right) = \frac{\delta Z}{Z}\left(1 + \frac{\chi}{\zeta}\right) \quad (79)$$

ISSN 2664-3413 (Print) 2664-3421 (Online)
Daffodil International University Journal of Business and Entrepreneurship, Vol. 14, No. 1, PP. 64-84, June, 2021

The relative variation in stock price is equivalent to relative variation in equity. As $\zeta \to 0$,

$\frac{\chi}{\zeta}$ becomes unbounded and the company will be highly leveraged. A company having a lot of debt in her capital structure is highly leveraged but where the debt tends to zero, the firm is unleveraged. Define volatilities as $\Phi_\eta = \Phi_\zeta = \Phi_Z \rho; \rho = \left(1 + \frac{\chi}{\zeta}\right)$ (80)

where $\Phi_\theta = \{\Phi_\eta, \Phi_\zeta, \Phi_Z\}$ are the volatilities of price, equity and asset value. The more leveraged a company is, the more the volatility of the stock are in relation to the aggregate of

the insurance portfolio. Since η is the market price of the equity at time ξ , then the relative change in price comes up as

$$\frac{\delta\eta}{\eta} = \mu\delta\xi + \Phi\delta W \Rightarrow \delta\eta = \mu\eta\delta\xi + \Phi\eta\delta W, \eta(0) = K \quad (81)$$

$$\delta \log \eta(\xi) = \frac{\delta\eta}{\eta} - 0.5\frac{\Phi^2\eta^2\delta\xi}{\eta^2} = \left(\mu - 0.5\Phi^2\right)\delta\xi + \Phi\delta W \quad (82)$$

$$\eta(\xi) = Ke^{\left(\mu - 0.5\Phi^2\right)\xi + \Phi W(\xi)} = KEXP\left(\left(\mu - \frac{1}{2}\omega^2\right)\xi + \Phi W(\xi)\right) \quad (83)$$

$\eta(0)$ is the initial price, $\mu > 0$ is the drift and ω is the volatility of equity. The implication is that if an investor buys equity at asking price $\eta_{ASKING}(\xi)$ and sells it at bidding price $\eta_{BIDDING}(\xi)$, then $\eta_{ASKING}(\xi) = \eta(\xi), \eta_{BIDDING}(\xi) = (1-\theta)\eta(\xi), 0 < \theta < 1$, where θ is the transaction cost. If β is the total sales and ρ is the total purchases of equity. If $\Sigma(\xi)$ is the amount of money invested in equity, then applying equation (81)

$$\delta\Sigma(\xi) = \mu\Sigma(\xi)d\xi + \Phi\Sigma(\xi)dW - (d\beta - d\rho)$$

6.1 Theorem 2

If, $\Phi_P = \Phi_\zeta = \Phi_Z \rho, \rho = \left(1 + \frac{\chi}{\zeta}\right)$ (84a)

then $\zeta \frac{d\Phi_Z \rho}{d\zeta} + \chi \frac{d\Phi_Z \rho}{d\chi} = 0 \Rightarrow -1 \le \Xi_\zeta \le 0 \le \Xi_\chi \le 1$ (84b)

6.1.1 Proof

$(i) \Rightarrow (ii)$

The elasticity of stock volatilities with respect to ζ and χ are defined by

$$\left[\Xi_{\zeta} = \frac{\frac{d\Phi_Z \rho}{d\zeta}}{\frac{\Phi_Z \rho}{\zeta}} = \frac{\frac{d}{d\zeta}\left[\Phi_Z\left(1+\frac{\chi}{\zeta}\right)\right]}{\frac{\Phi_Z \rho}{\zeta}} = \frac{-\Phi_Z \chi \zeta^{-2}}{\frac{\Phi_Z\left(1+\frac{\chi}{\zeta}\right)}{\zeta}} = \frac{-\Phi_Z \chi \zeta^{-2}}{1} \times \frac{\zeta}{\Phi_Z\left(1+\frac{\chi}{\zeta}\right)}\right] \quad (85)$$

$$\left[\Xi_{\zeta} = \frac{-\Phi_Z \chi \zeta^{-2}}{1} \times \frac{\zeta}{\Phi_Z\left(\frac{\zeta+\chi}{\zeta}\right)} = -\frac{\Phi_{\zeta}}{\left(\frac{\zeta+\chi}{\zeta}\right)} \chi\zeta^{-2} \times \frac{\zeta^2}{\frac{\Phi_{\zeta}}{\left(\frac{\zeta+\chi}{\zeta}\right)}(\zeta+\chi)}\right] \quad (86)$$

$$\left[\Xi_{\zeta} = \frac{-\frac{\Phi_{\zeta}}{1}\frac{\zeta}{(\zeta+\chi)}\chi\zeta^{-2}}{1} \times \frac{\zeta^2}{\frac{\Phi_{\zeta}}{1}(\zeta+\chi)\frac{\zeta}{(\zeta+\chi)}} = -\frac{\Phi_{\zeta}}{1}\frac{\zeta}{(\zeta+\chi)}\chi\zeta^{-2} \times \frac{\zeta^2}{\zeta\Phi_{\zeta}} = -\frac{\chi}{(\zeta+\chi)}\right] \quad (87)$$

$$\Xi_{\zeta} = -\frac{\chi}{(\zeta+\chi)} \quad (88)$$

$$\Xi_{\chi} = \frac{\frac{d\Phi_Z \rho}{d\chi}}{\frac{\Phi_Z \rho}{\chi}} = \frac{\frac{d}{d\chi}\left[\Phi_Z\left(1+\frac{\chi}{\zeta}\right)\right]}{\frac{\Phi_Z \rho}{\chi}} = \frac{\Phi_Z \zeta^{-1}}{\frac{\Phi_Z\left(1+\frac{\chi}{\zeta}\right)}{\chi}} = \frac{\Phi_Z \zeta^{-1}}{1} \times \frac{\chi}{\Phi_Z\left(1+\frac{\chi}{\zeta}\right)} \quad (89)$$

$$\Xi_{\chi} = \frac{\frac{d\Phi_Z \rho}{d\chi}}{\frac{\Phi_Z \rho}{\chi}} = \frac{\frac{d}{d\chi}\left[\Phi_Z\left(1+\frac{\chi}{\zeta}\right)\right]}{\frac{\Phi_Z \rho}{\chi}} = \frac{\Phi_Z \zeta^{-1}}{\frac{\Phi_Z\left(1+\frac{\chi}{\zeta}\right)}{\chi}} = \frac{\Phi_Z \zeta^{-1}}{1} \times \frac{\chi}{\Phi_Z\left(1+\frac{\chi}{\zeta}\right)} \quad (90)$$

$$\Xi_{\chi} = \frac{\Phi_Z \zeta^{-1}}{1} \times \frac{\chi}{\Phi_Z \left(\frac{\zeta + \chi}{\zeta}\right)} = \frac{\Phi_Z \zeta^{-1}}{1} \times \frac{\chi\zeta}{\Phi_Z (\zeta + \chi)} = \frac{\chi}{(\zeta + \chi)} \tag{91}$$

$$\Xi_{\chi} = \frac{\chi}{(\zeta + \chi)} < 1 \tag{92}$$

$$-\Xi_{\zeta} = \Xi_{\chi} \Rightarrow \Xi_{\zeta} + \Xi_{\chi} = 0 \tag{93}$$

$$\frac{\zeta}{\Phi_Z \rho} \frac{d\Phi_Z \rho}{d\zeta} + \frac{\chi}{\Phi_Z \rho} \frac{d\Phi_Z \rho}{d\chi} = 0 \Rightarrow \zeta \frac{d\Phi_Z \rho}{d\zeta} + \chi \frac{d\Phi_Z \rho}{d\chi} = 0 \tag{94}$$

$$\Xi_{\rho} = \frac{\frac{d\Phi_Z \rho}{d\rho}}{\frac{\Phi_Z \rho}{\rho}} = \frac{\Phi_Z}{1} \times \frac{\rho}{\Phi_Z \rho} = 1 \tag{95}$$

Consequently, the interval of validity of the elasticities of debt and equity is
$-1 \le \Xi_{\zeta} \le 0 \le \Xi_{\chi} \le 1$

7. Discussion of Results

We introduced a stochastic partial differential equation to analyze the development of investment process in portfolio choice. The differential equations of portfolio value are derived in two formulations leading to equilibrium and non-equilibrium conditions. Furthermore, applying the property of Wiener process, we computed the leverage effect function. The novel element in this paper is that the *relative* random change in asset value is the volatility process resulting in the cancellation of the elasticity of debt and equity and interval of validity of elasticities. The analytical correspondence between both equity & volatility and equity & returns was examined in a leveraged company. Everything else assumed constant in an insurance company with equity and default-free debt in its capital structure, the elasticity of stock volatility with respect to equity functionally depends on insurance

company's leverage defined as $\Xi_{\zeta} = -\frac{\chi}{(\zeta + \chi)}$. Furthermore, the elasticity of stock volatility in relation to debt is dependent on company's leverage given by $\Xi_{\chi} = \frac{\chi}{(\zeta + \chi)}$

It is clear from equations (92) and (95) that $0 \le \Xi_{\chi} \le \Xi_{\rho} \Rightarrow 0 \le \Xi_{\chi} \le 1$ (96)

Furthermore, $0 \le -\Xi_{\zeta} \le \Xi_{\rho} \Rightarrow 0 \ge \Xi_{\zeta} \ge -\Xi_{\rho} \Rightarrow 0 \ge \Xi_{\zeta} \ge -1$ (97)

$$0 \le \Xi_{\chi} \le \Xi_{\rho} \Rightarrow 0 \le \Xi_{\chi} \le 1 \tag{98}$$

$$0 \le -\Xi_{\zeta} \le \Xi_{\rho} \Rightarrow 0 \ge \Xi_{\zeta} \ge -\Xi_{\rho} \tag{99}$$

$$0 \le -\Xi_\zeta \le \Xi_\rho \Rightarrow 0 \ge \Xi_\zeta \ge -1 \tag{100}$$

$$0 \le -\Xi_\zeta \le \Xi_\rho \Rightarrow -1 \le \Xi_\zeta \le 0 \tag{101}$$

Consequently, $-1 \le \Xi_\zeta \le 0 \le \Xi_\chi \le 1$ (102)

putting equation (81) in (79), we have

$$\eta\mu\delta\xi + \eta\omega\delta W = \frac{\eta\delta Z}{Z}\left(1+\frac{\chi}{\zeta}\right) = \frac{\eta\delta Z}{Z}\times\rho \tag{103}$$

$$\zeta\mu\delta\xi + \zeta\omega\delta W = \frac{\zeta\delta Z}{Z}\left(1+\frac{\chi}{\zeta}\right) = \frac{\zeta\delta Z}{Z}\times\rho \tag{104}$$

The theoretical value of the magnitude of the leverage effect is unity, thus the financial leverage assumes that elasticity of equity volatility is negatively correlated to equity returns, this is because when the price of a stock drops, then stock volatility rises accounting for the correspondence between expected returns and volatility of stock and consequently volatility and stock returns are negatively correlated. Financial experts observe that when there is a decline in stock price, then financial leverage increases resulting in stock return volatility. Future study can be geared towards the numerical solution of the PDE's and empirical result where there is availability of financial data for the analysis of model parameters. Before concluding this section, in the diffusion equation $\frac{\partial Z(\eta,\xi)}{\partial \xi} = \mathrm{rP}\frac{\partial^2 Z(\eta,\xi)}{\partial \eta^2}$

$\frac{\partial Z(\eta,\xi)}{\partial \xi}$ and $\frac{\partial^2 Z(\eta,\xi)}{\partial \eta^2}$ are proportional. We can then set $\mathrm{rP}=\frac{\beta^2}{2}$ so that

$\frac{\partial Z(\eta,\xi)}{\partial \xi} = \frac{\beta^2}{2}\frac{\partial^2 Z(\eta,\xi)}{\partial \eta^2}$ to obtain a Brownian motion and consequently β becomes the volatility of equity. Again, we can set $Z(\eta,\xi) = probability = \vartheta(\eta,\xi)$ so that $\frac{\partial \vartheta(\eta,\xi)}{\partial \xi} - \frac{\beta^2}{2}\frac{\partial^2 \vartheta(\eta,\xi)}{\partial \eta^2} = 0$. This differential equation thus explains the effect of the instantaneous intensity of uncertainty under Brownian motion and the ways it will affect stock values of insurance portfolio.

8. Conclusion

Financial market randomness and uncertainties account for constructing models for efficiently approximating the value of managed portfolios. Optimal management of portfolio under stochastic framework is a significant force in allocating limited resources among variable assets in order that an investor can optimize satisfaction from terminal wealth. Continuous time financial model such as optimal differential equation was adopted to specify portfolio dynamics of stocks and derivatives. In this paper, it has been demonstrated how to measure the value of a portfolio through second order partial differential equation under homogeneous and non-homogeneous framework. Furthermore, it also investigates how the leverage effect function can be evaluated under Brownian motion.

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